Problema 268, propuesto por D.M.Batinetzu – Giurgiu, Bucarest, y N.Stanciu, Buzau, Romania
Si $m, n, x, y, z > 0$, entonces

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\cdot\left(\left(\frac{xy}{my+nz}+\frac{yz}{mz+nx}+\frac{zx}{mx+ny}\right)\right)\geq\frac{3}{m+n}+3\sqrt[3]{\left(\frac{(x+y)(y+z)(z+x)}{(mx+ny)\cdot(my+nz)\cdot(mz+nx)}\right)}\ .$$

Solution (proposed by **Elena Codeci, Daniel Codeci** and **Daniel Văcaru**)
One find

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\cdot\left(\frac{xy}{my+nz}+\frac{yz}{mz+nx}+\frac{zx}{mx+ny}\right)=\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\left(\frac{xy}{my+nz}\right)+\frac{xy}{z\cdot(my+nx)}+\left(\frac{1}{y}+\frac{1}{z}\right)\cdot\left(\frac{yz}{mz+nx}\right)+\left(\frac{1}{x}\right)\cdot\left(\frac{yz}{mz+nx}\right)+$$
$$+\left(\frac{1}{x}+\frac{1}{z}\right)\cdot\left(\frac{zx}{mx+ny}\right)+\frac{1}{y}\cdot\left(\frac{zx}{mx+ny}\right).$$

One may write

$$\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\left(\frac{xy}{my+nz}\right)+\left(\frac{1}{y}+\frac{1}{z}\right)\cdot\left(\frac{yz}{mz+nx}\right)+\left(\frac{1}{x}+\frac{1}{z}\right)\cdot\left(\frac{zx}{mx+ny}\right)=\frac{x+y}{my+nz}+\frac{y+z}{mz+nx}+\frac{x+z}{mx+ny}=\frac{\left[\sqrt{(x+y)}\right]^2}{my+nz}+\frac{\left[\sqrt{(y+z)}\right]^2}{mz+nx}+\frac{\sqrt{\left[(x+z)\right]^2}}{mx+ny}$$
$$\overbrace{\geq}^{Bergstrom}\geq\frac{\left[\sqrt{(x+y)}+\sqrt{(y+z)}+\sqrt{(z+x)}\right]^2}{(my+nz)+(mz+nx)+(mx+ny)}\overbrace{\geq}^{AM-GM}\frac{3\cdot\left[\sqrt{(x+y)}\cdot\sqrt{(y+z)}+\sqrt{(y+z)}\sqrt{(z+x)}+\sqrt{(z+x)}\sqrt{(x+y)}\right]}{(m+n)\cdot(x+y+z)}\geq$$
$$\geq\frac{3\cdot\cancel{(x+y+z)}}{(m+n)\cdot\cancel{(x+y+z)}}=\frac{3}{m+n}$$

The remaining sum can be write as

$$\frac{xy}{z\cdot(my+nx)}+\frac{yz}{x\cdot(mz+nx)}+\frac{zx}{y\cdot(mx+ny)}\underbrace{\geq}_{AM-GM}3\sqrt[3]{\left(\left(\frac{xy}{z\cdot(my+nx)}\right)\left(\frac{yz}{x\cdot(mz+nx)}\right)\left(\frac{zx}{y\cdot(mx+ny)}\right)\right)}=3\sqrt[3]{\left(\frac{xyz}{(my+nx)\cdot(mz+ny)\cdot(mx+nz)}\right)}\ .$$

That concludes this solution.

Observation

We think that one better inferior bound could be obtained if one write

$$\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\left(\frac{xy}{my+nz}\right)+\left(\frac{1}{y}+\frac{1}{z}\right)\cdot\left(\frac{yz}{mz+nx}\right)+\left(\frac{1}{x}+\frac{1}{z}\right)\cdot\left(\frac{zx}{mx+ny}\right)=\frac{x+y}{my+nz}+\frac{y+z}{mz+nx}+\frac{x+z}{mx+ny}=\frac{\left[\sqrt{(x+y)}\right]^2}{my+nz}+\frac{\left[\sqrt{(y+z)}\right]^2}{mz+nx}+\frac{\sqrt{\left[(x+z)\right]^2}}{mx+ny}$$
$$\overbrace{\geq}^{Bergstrom}\geq\frac{\left[\sqrt{(x+y)}+\sqrt{(y+z)}+\sqrt{(z+x)}\right]^2}{(my+nz)+(mz+nx)+(mx+ny)}=\frac{2x+2y+2z+2\cdot\sqrt{(x+y)}\cdot\sqrt{(y+z)}+2\cdot\sqrt{(y+z)}\sqrt{(z+x)}+2\cdot\sqrt{(z+x)}\sqrt{(x+y)}}{(m+n)\cdot(x+y+z)}\geq.$$
$$\geq\frac{4\cdot\cancel{(x+y+z)}}{(m+n)\cdot\cancel{(x+y+z)}}=\frac{4}{m+n}$$

It follows that

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\cdot\left(\left(\frac{xy}{my+nz}+\frac{yz}{mz+nx}+\frac{zx}{mx+ny}\right)\right)\geq\frac{4}{m+n}+3\sqrt[3]{\left(\frac{(x+y)(y+z)(z+x)}{(mx+ny)\cdot(my+nz)\cdot(mz+nx)}\right)}\ .$$