PROPOSED SOLUTION TO PROBLEMA 141

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Sea $\binom{n}{i}$ un numero combinatorio, $n, i \in N$ $n \geq i$. Demostrar entonces que

$$\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}$$

y $e^{\frac{1}{2}n^2}$ son dos infinitos asintoticamente equivalentes, es decir,

$$\lim_{n\to\infty} \frac{e^{\frac{1}{2}n^2}}{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}} = 1.$$

Solution. In what follows we are going to prove that

$$\lim_{n\to\infty} \frac{e^{\frac{1}{2}n^2}}{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}} = \infty.$$

We need the following Lemma.

Lemma 0.1. *Glaisher-Kinkelin constant. The following limit holds:*

$$A = \lim_{n\to\infty} \frac{1^1 2^2 \cdots n^n}{n^{n^2/2+n/2+1/12} e^{-n^2/4}},$$

where $A = 1.282712$ *is the Glaisher-Kinkelin constant.*

Let $x_n = \frac{e^{\frac{1}{2}n^2}}{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}$. We have

$$\ln x_n = \frac{n^2}{2} - \sum_{k=1}^{n} \ln \frac{n!}{k!(n-k)!} = \frac{n^2}{2} - (n+1)\ln n! + 2\sum_{k=1}^{n} \ln k!.$$

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On the other hand, we note that

$$\sum_{k=1}^{n} \ln k! = n \ln 1 + (n-1)\ln 2 + \cdots + 2\ln(n-1) + \ln n = \sum_{k=1}^{n}(n+1-k)\ln k$$

$$= (n+1)\ln n! - \sum_{k=1}^{n} k \ln k.$$

Thus,

$$\begin{aligned} \ln x_n &= \frac{n^2}{2} + (n+1)\ln n! - 2\sum_{k=1}^{n} k\ln k \\ &= \frac{n^2}{2} + (n+1)\ln n! - 2\left(\sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln n + \frac{n^2}{4}\right) \\ &- \left(n^2 + n + \frac{1}{6}\right)\ln n + \frac{n^2}{2} = n^2 + (n+1)\ln n! - \left(n^2 + n + \frac{1}{6}\right)\ln n - 2A_n, \end{aligned}$$

where

$$A_n = \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln n + \frac{n^2}{4}.$$

Using *Stirling's formula* $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we get that

$$n^2 + (n+1)\ln n! - \left(n^2 + n + \frac{1}{6}\right)\ln n \approx (n+1)\ln\sqrt{2\pi n} - n - \frac{\ln n}{6}.$$

It follows, based on Lemma 0.1, that

$$\ln x_n \approx (n+1)\ln\sqrt{2\pi n} - n - \frac{\ln n}{6} - 2A_n \to \infty - 2\ln A = \infty,$$

and hence $x_n \to \infty$.

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