

**PROPOSED SOLUTION TO PROBLEMA 143, VOL 29,
JANUARY-FEBRUARY 2007**

Sea $l \geq 0$ un número natural. Calcular

$$\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+2l+1) - \gamma \right),$$

siendo γ la constante de *Euler-Mascheroni*.

Solution. The series equals

$$\frac{\gamma}{2} - (2l+1) \ln 2 + \frac{1}{2} \ln \pi - \ln \frac{(l!)^2}{(2l+1)!}.$$

Let $H_m = \sum_{n=1}^m \frac{1}{n}$ be the m^{th} harmonic number. The first step is to prove the convergence of the series, so we break the series as

$$\sum_{n=1}^{\infty} (-1)^n (H_n - \ln n - \gamma) + \sum_{n=1}^{\infty} (-1)^n (\ln n - \ln(n+2l+1)),$$

and show that both series are *Leibniz* series.

The first series. We have that $H_n - \ln n - \gamma \searrow 0$ and $H_n - \ln n$ decreases with n . Indeed

$$H_{n+1} - \ln(n+1) < H_n - \ln n, \text{ for all } n \geq 1, \text{ since } \frac{1}{n+1} < \ln \frac{n+1}{n}.$$

The second series. We have, based on straightforward calculations, that

$$\lim_{n \rightarrow \infty} (\ln n - \ln(n+2l+1)) = 0 \quad \text{and} \quad 0 > \ln(n+1) - \ln(n+2l+2) > \ln n - \ln(n+2l+1).$$

The sum of the series. Let S_{2m} be the $2m^{\text{th}}$ partial sum of the series. We have

$$(0.1) \quad S_{2m} = \sum_{n=1}^{2m} (-1)^n (H_n - \ln(n+1+2l) - \gamma) = \frac{1}{2} H_m - \sum_{n=1}^{2m} (-1)^n \ln(n+1+2l).$$

Also, if $\gamma_m = H_m - \ln m$, then $\lim_{m \rightarrow \infty} \gamma_m = \gamma$. A calculation shows that

$$(0.2) \quad \sum_{n=1}^{2m} (-1)^n \ln(n+1+2l) = \ln \left(\frac{(2m+1+2l)!}{2^{2m}((m+l)!)^2} \cdot \frac{(l!)^2}{(2l+1)!} \right).$$

The *Stirling's* formula, $m! = \sqrt{2\pi m} \left(\frac{m}{e}\right)^m (1 + O(m^{-1}))$, and the *Taylor's* formula, $\ln(1+x) = x - \frac{x^2}{2} + O(x^{-3})$, for $|x| < 1$, yield (the straightforward calculations are left to the reader):

$$(0.3) \quad \ln \frac{(2m+1+2l)!}{2^{2m}((m+l)!)^2} = \frac{1}{2} \ln m + (2l+1) \ln 2 - \frac{1}{2} \ln \pi + O(m^{-1}).$$

Combining (0.1), (0.2) and (0.3), we get that

$$S_{2m} = \frac{1}{2} \gamma_m - (2l+1) \ln 2 + \frac{1}{2} \ln \pi - \ln \frac{(l!)^2}{(2l+1)!} - O(m^{-1}),$$

whose limit, as $m \rightarrow \infty$, shows that the desired result yields.

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Revista Escolar de la Olimpiada Iberoamericana de Matemática

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