PMN43-1

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ABC é un triángulo rectángulo en $B.$ H é o pé da altura desde $B.$ As medianas que parten de A e $B,$ relativas respectivamente aos lados BH e HC dos triángulos ABH e $HBC,$ córtanse no punto $M.$ As bisectrices interiores dos ángulos BAH e HBC córtanse no punto $N.$ Se $\alpha = \measuredangle AMN,$ $\beta = \measuredangle ANH,$ determinar o ángulo $\measuredangle BHN.$

Solución por Bruno Salgueiro Fanego, Viveiro, Lugo.

Sexan P e Q os respectivos puntos medios de BH e $HC,$ R o punto de intersección de BH e $AN,$ S o punto de intersección de CA e $BN,$ $\measuredangle A = \measuredangle BAC$ e $\gamma = \measuredangle BHN.$ Na referencia de orixe H na que se teñen as coordenadas $A(2a,0),$ $C(-2c,0),$ $B(0,2b),$ resulta que $P(0,b),$ $Q(-c,0)$ e $0 = \overrightarrow{BC} \cdot \overrightarrow{BA} = (-2c,-2b)\cdot(2a,-2b) = -4ca + 4b^2,$ logo $b = \sqrt{ca},$ sendo entón M a intersección das rectas $AP : y = -\dfrac{b}{2a}x + b$ e $BQ : y = \dfrac{2b}{c}x + 2b,$ que é $M\left(\dfrac{-2ca}{c+4a}, \dfrac{2\sqrt{ca}\,(c+2a)}{c+4a}\right),$ co cal $AM \perp BM,$ pois

$$\overrightarrow{MA}\cdot\overrightarrow{MB} = \left(\frac{4a(c+2a)}{c+4a}, \frac{-2\sqrt{ca}\,(c+2a)}{c+4a}\right)\cdot\left(\frac{2ca}{c+4a}, \frac{4a\sqrt{ca}}{c+4a}\right) = \frac{8ca^2(c+2a)}{c+4a} - \frac{8ca^2(c+2a)}{c+4a} = 0.$$

Como $\measuredangle HRA = \pi - (\measuredangle RAH + \measuredangle AHR) = \pi - \left(\dfrac{\measuredangle A}{2} + \measuredangle AHR\right) = \dfrac{\pi - \measuredangle A}{2},$ tense que

$$\measuredangle NRH = \pi - \measuredangle HRA = \frac{\pi + \measuredangle A}{2} \quad \text{e}$$

$$\gamma = \measuredangle RHN = \pi - (\measuredangle RNH + \measuredangle NRH) = \pi - (\measuredangle ANH + \measuredangle NRH) = \pi - \left(\beta + \frac{\pi + \measuredangle A}{2}\right) = \frac{\pi - \measuredangle A}{2} - \beta.$$

Nótese que $\measuredangle NBA = \measuredangle SBA = \measuredangle SBH + \measuredangle HBA = \dfrac{\measuredangle A}{2} + \left(\dfrac{\pi}{2} - \measuredangle A\right) = \dfrac{\pi - \measuredangle A}{2}$ implica que $\measuredangle ANB = \pi - (\measuredangle NBA + \measuredangle BAN) = \pi - \left(\dfrac{\pi - \measuredangle A}{2} + \dfrac{\measuredangle A}{2}\right) = \dfrac{\pi}{2},$ co cal $AN \perp BN,$ de onde resulta, ao ser $AM \perp BM,$ que $\measuredangle NBA = \measuredangle SBA = \measuredangle SBH + \measuredangle HBA = \dfrac{\measuredangle A}{2} + \left(\dfrac{\pi}{2} - \measuredangle A\right) = \dfrac{\pi - \measuredangle A}{2}$ e $\measuredangle NMA = \alpha$ son iguais, por ser ángulos inscritos na circunferencia de diámetro AB que abarcan o mesmo arco, $AN,$ é dicir, que $\gamma = \dfrac{\pi - \measuredangle A}{2} - \beta = \alpha - \beta.$