Problemas de nivel medio e de olimpíadas 54.

1. Sexan x e y números reais positivos satisfacendo as ecuacións $x^2+y^2=1$ e $x^4+y^4=\frac{17}{18}$.

Calcule o valor de $\frac{1}{xy}$.

Solución enviada por Bruno Salgueiro Fanego, Viveiro, Lugo.

Se x e y son números complexos calesquera,

$$\begin{cases} x^2+y^2=1 \\ x^4+y^4=\frac{17}{18} \end{cases} \Leftrightarrow \begin{cases} y^2=1-x^2 \\ x^4+\left(1-x^2\right)^2=\frac{17}{18} \end{cases} \Leftrightarrow \begin{cases} y^2=1-x^2 \\ 36x^4-36x^2+1=0 \end{cases} \Leftrightarrow \begin{cases} x^2=\frac{\left(\sqrt{2}\pm 1\right)^2}{6} \\ y^2=1-x^2 \end{cases}$$

$$\Leftrightarrow \left(x^2,y^2\right)\in\left\{\left(\frac{\left(\sqrt{2}+1\right)^2}{6},\frac{\left(\sqrt{2}-1\right)^2}{6}\right),\left(\frac{\left(\sqrt{2}-1\right)^2}{6},\frac{\left(\sqrt{2}+1\right)^2}{6}\right)\right\}$$

$$\Leftrightarrow (x,y)\in\left\{\left(\frac{\sqrt{2}+1}{\sqrt{6}},\frac{\sqrt{2}-1}{\sqrt{6}}\right),\left(\frac{\sqrt{2}+1}{\sqrt{6}},-\frac{\sqrt{2}-1}{\sqrt{6}}\right),\left(-\frac{\sqrt{2}+1}{\sqrt{6}},\frac{\sqrt{2}-1}{\sqrt{6}}\right),\left(-\frac{\sqrt{2}+1}{\sqrt{6}},-\frac{\sqrt{2}-1}{\sqrt{6}}\right),\right.$$

$$\left.\left(\frac{\sqrt{2}-1}{\sqrt{6}},\frac{\sqrt{2}+1}{\sqrt{6}}\right),\left(\frac{\sqrt{2}-1}{\sqrt{6}},-\frac{\sqrt{2}+1}{\sqrt{6}}\right),\left(-\frac{\sqrt{2}-1}{\sqrt{6}},\frac{\sqrt{2}+1}{\sqrt{6}}\right),\left(-\frac{\sqrt{2}-1}{\sqrt{6}},-\frac{\sqrt{2}+1}{\sqrt{6}}\right)\right\}$$

$$\Rightarrow xy=\pm\frac{1}{6}\Rightarrow\frac{1}{xy}=\pm 6.$$

Por tanto, no caso de que x e y sexan números reais positivos, $\frac{1}{xy}=6$.

Observación:

$$4x^2y^2+4xy+1=(2xy+1)^2=\left(x^2+2xy+y^2\right)^2=\left[(x+y)^2\right]^2=(x+y)^4=\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k}$$

$$=x^4+4x^3y+6x^2y^2+4xy^2+y^4=6x^2y^2+4xy\left(x^2+y^2\right)+x^4+y^4=6x^2y^2+4xy\cdot 1+\frac{17}{18}$$

$$\Rightarrow 4x^2y^2+4xy+1=6x^2y^2+4xy+\frac{17}{18}\Rightarrow\frac{1}{18}=2x^2y^2\Rightarrow(xy)^2=\frac{1}{36}.$$

Cando x e y son números reais positivos, $xy=\frac{1}{6}$ e, polo tanto, $\frac{1}{xy}=6$.