Proposed solution of problem 295, 58 (2018)

Dear Editor of "Rivista Escolar de la Olimpiada Iberoamericana de Matemática", I would like to submit the following solution of problem 295

Sean $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ *sucesiones de números reales positivos tale que* $b_n = a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdots \sqrt[n]{a_n}$ *y* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n} = a$. *Calcular* $\lim_{n\to\infty} \left(\frac{(n+1)^2}{b_{n+1}} - \frac{n^2}{\sqrt[n]{b_n}} \right)$

As stated the limit equals $-\infty$.

By definition, there exists n' such that $n > n'$ implies $2an \geq \frac{a_{n+1}}{a_n} > na/2$. This in turn implies

$$\frac{a_{n+1}}{a_{n'}} = \frac{a_{n+1}}{a_n} \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{n'+1}}{a_{n'}} \geq \left(\frac{a}{2}\right)^{n+1-n'} \frac{n!}{(n'-1)!}$$

and

$$\frac{a_{n+1}}{a_{n'}} \leq (2a)^{n+1-n'} \frac{n!}{(n'-1)!}$$

We get

$$b_n = \underbrace{a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdots \sqrt[n']{a_{n'}}}_{\doteq A} \prod_{k=n'+1}^{n} \left(\frac{a}{2}\right)^{1+\frac{1}{k}-\frac{n'}{k}} \prod_{k=n'+1}^{n} \frac{k!^{\frac{1}{k}}}{((n'-1)!)^{\frac{1}{k}}}$$

We know that $C_1 \ln n \leq \sum_{k=k_0}^{n} \frac{1}{k} \leq C_2 \ln n$ where C_1, C_2 depends of k_0. Thus

$$\prod_{k=n'+1}^{n} \left(\frac{a}{2}\right)^{1+\frac{1}{k}-\frac{n'}{k}} \geq \left(\frac{a}{2}\right)^{n-n'+C(1-n')\ln n} \doteq B_n$$

where $C = C_1$ if $a < 1$ and $C = C_2$ otherwise.

$$\prod_{k=n'+1}^{n} \frac{1}{((n'-1)!)^{\frac{1}{k}}} \geq \frac{1}{(n'-1)^{C_2 \ln n}} \doteq C_n$$

$$\prod_{k=n'+1}^{n} k!^{\frac{1}{k}} \geq \prod_{k=n'+1}^{n} k = \frac{n!}{n'!}$$

The consequence is

$$\frac{(n+1)^2}{b_{n+1}} \leq \frac{(n+1)^2(n'!)}{AB_nC_nn!} \to 0$$

On the other hand it is not difficult to show that $n^2/\sqrt[n]{b_n} \to +\infty$ so the whole limit would be equal to $-\infty$. May be the true limit is

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}}\right) \tag{1}$$

To prove the divergence of $n^2/\sqrt[n]{b_n}$ we employ Cesaro–Stolz theorem by writing

$$\frac{n^2}{\sqrt[n]{b_n}} = \frac{(n^{2n})^{\frac{1}{n}}}{(b_n)^{\frac{1}{n}}} \tag{2}$$

so we come to the study of the limit

$$\lim_{n\to\infty}\frac{(n+1)^{2n}(n+1)^2}{n^{2n}}\frac{b_n}{b_{n+1}} = \lim_{n\to\infty} e^2\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}$$

We know that

$$\prod_{k=n'+1}^{n}(2a)^{1+\frac{1}{k}-\frac{n'}{k}} \leq (2a)^{n-n'+C'(1-n')\ln n} \doteq B'_n$$

where $C = C_1$ if $a > 1$ and $C = C_2$ and

$$\prod_{k=n'+1}^{n}\frac{1}{((n'-1)!)^{\frac{1}{k}}} \leq \frac{1}{(n'-1)^{C_1\ln n}} \doteq C'_n$$

It follows

$$\frac{(n+1)^2}{b_{n+1}} \geq \frac{(n+1)^2}{\sqrt[n+1]{AB'_nC'_nn!}}$$

Now

$$\sqrt[n+1]{AB'_nC'_n} \to 2a$$

while

$$\frac{(n+1)^2}{\sqrt[n+1]{n!}} \to +\infty$$

and this may be seen by using Cesaro–Stolz again.

$$\frac{(n+1)^2}{\sqrt[n+1]{n!}} = \left(\frac{(n+1)^{2(n+1)}}{n!}\right)^{\frac{1}{n+1}}$$

and we consider

$$\frac{(n+2)^{2(n+2)}}{(n+1)^{2(n+1)}}\frac{(n+1)!}{n!} \to +\infty$$

The divergence of the quality in (2) is proven.

Now we come back to (1) by writing

$$\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}}\right) = \frac{n^2}{n\sqrt[n]{b_n}} \frac{\frac{(n+1)^2}{n^2}\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} - 1}{\ln\left(\frac{(n+1)^2}{n^2}\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)} \ln\left(\frac{(n+1)^2}{n^2}\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)^n \tag{3}$$

To compute the limit of (3) we employ Cesaro–Stolz.

First limit.

$$\frac{n^2}{n\sqrt[n]{b_n}} = \left(\frac{n^n}{b_n}\right)^{1/n}$$

$$\frac{(n+1)^{n+1}}{n^n}\frac{b_n}{b_{n+1}} = \underbrace{\frac{(n+1)^n}{n^n}}_{\to e}\frac{n+1}{\sqrt[n+1]{a_{n+1}}}$$

Moreover

$$\frac{n+1}{\sqrt[n+1]{a_{n+1}}} = \left(\frac{(n+1)^{n+1}}{a_{n+1}}\right)^{\frac{1}{n+1}}$$

and then we pass to

$$\frac{(n+2)^{n+2}}{(n+1)^{n+1}}\frac{a_{n+1}}{a_{n+2}} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}}\frac{(n+2)a_{n+1}}{a_{n+2}} \to \frac{e}{a} \implies \left(\frac{n^n}{b_n}\right)^{1/n} \to \frac{e}{a} \tag{4}$$

Second limit.

$$\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} = \frac{\sqrt[n]{b_n}}{\sqrt[n]{b_{n+1}}} b_{n+1}^{\frac{1}{n(n+1)}}$$

$$(AB_nC_nn!)^{\frac{1}{n(n+1)}} \le b_{n+1}^{\frac{1}{n(n+1)}} \le (AB'_nC'_nn!)^{\frac{1}{n(n+1)}} \implies b_{n+1}^{\frac{1}{n(n+1)}} \to 1$$

As for $\dfrac{\sqrt[n]{b_n}}{\sqrt[n]{b_{n+1}}}$ we pass to

$$\frac{b_{n+1}}{b_{n+2}}\frac{b_{n+1}}{b_n} = \frac{a_{n+1}^{\frac{1}{n+1}}}{a_{n+2}^{\frac{1}{n+2}}} = \frac{a_{n+1}^{\frac{1}{n+1}}}{a_{n+2}^{\frac{1}{n+1}}} \underbrace{a_{k+2}^{\frac{1}{(n+1)(n+2)}}}_{\to 1}$$

Again Cesaro–Stolz for $\lim\limits_{n\to\infty} \dfrac{a_{n+1}^{\frac{1}{n+1}}}{a_{n+2}^{\frac{1}{n+1}}}$

$$\frac{a_{k+2}}{a_{k+3}}\frac{a_{k+2}}{a_{k+1}} = \frac{ka_{k+2}}{a_{k+3}}\frac{a_{k+2}}{ka_{k+1}} \to 1$$

This implies that $\dfrac{\sqrt[n]{b_n}}{\sqrt[n]{b_{n+1}}} \to 1$ and then

$$\frac{\frac{(n+1)^2}{n^2}\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} - 1}{\ln\left(\frac{(n+1)^2}{n^2}\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)} \to 1 \tag{5}$$

Third limit.

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{n^2}\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)^n \doteq (6)$$

$$\left(\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}\right)^n = \frac{b_n}{b_{n+1}^{\frac{n}{n+1}}} = \frac{b_{n+1}^{\frac{1}{n+1}}}{a_{n+1}^{\frac{1}{n+1}}}$$

thus (Cesaro–Stolz)

$$\frac{b_{n+2}}{nb_{n+1}}\underbrace{\frac{na_{n+1}}{a_{n+2}}}_{\to 1/a} = \left(\frac{a_{n+2}}{n^{n+2}}\right)^{\frac{1}{n+2}}\frac{na_{n+1}}{a_{n+2}}$$

Again Cesaro–Stolz

$$\frac{a_{n+3}}{na_{n+2}}\frac{n^{n+2}}{(n+1)^{n+2}} \to \frac{a}{e} \implies (6) = e^2\frac{a}{e}\frac{1}{a} = e$$

By multiplying (4), (5) and (6) we get

$$\frac{e}{a} \cdot 1 \cdot \ln e = \frac{e}{a}$$

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Thank you, Best regards
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